13/10/15

Συνήθης γραμμική εξίσωση α' τάξης:

$$a_1(x)y'(x) + a_0(x)y(x) = b(x) \quad , x \in I$$

$a_1, a_0, b \in C(I)$, $I \subseteq \mathbb{R}$, I διάστημα

$a_1(x) \neq 0$, $x \in I$

$$y'(x) + \frac{a_0(x)}{a_1(x)} y(x) = \frac{b(x)}{a_1(x)}$$

(E). $y'(x) + p(x)y(x) = q(x)$, $p, q \in C(I)$

y λύση της (E) αν : y παραγωγίσιμη στο $I^{(*)}$ και ικανοποιεί την (E)

$\rightarrow$ $y'(x) = q(x)$, $x \in I \rightsquigarrow y(x) = c + \int q(x)dx$

$$y(x) - y(x_0) = \int_{x_0}^{x} q(s)ds \Rightarrow y(x) = y(x_0) + \int_{x_0}^{x} q(s)ds$$

$\rightarrow$ $y'(x) = -p(x)y(x) \Rightarrow \frac{y'(x)}{y(x)} = -p(x)$

$$\int_{x_0}^{x_0} \frac{y'(s)}{y(s)} ds = -\int_{x_0}^{x} p(s)ds \Rightarrow \ln|y(x)| - \ln|y(x_0)| = -\int_{x_0}^{x} p(s)ds$$

A.14(ii)

$y' + y\sqrt{x}\sin x = 0$, $x \geq 0$

$y'(x) = -y(x)\sqrt{x}\sin x$

$$\frac{y'(x)}{y(x)} = -\sqrt{x}\sin x \Rightarrow \int_{x_0}^{x} \frac{y'(s)}{y(s)} ds = -\int_{x_0}^{x} \sqrt{s}\sin s\, ds$$

$$\Rightarrow \ln|y(x)| = \ln|y(x_0)| - \int_{x_0}^{x} \sqrt{s}\sin s\, ds$$

$$y(x) = \pm e^{\ln|y(x_0)| - \int_{x_0}^{x} \sqrt{s}\sin s\, ds}$$

A.14(i) $xy' - y = x^2 e^x$, $I = (0, +\infty)$ ή $I = (-\infty, 0)$

$$\frac{xy' - y}{x^2} = e^x \Rightarrow \left(\frac{y}{x}\right)' = e^x$$

$$\Rightarrow \frac{y(x)}{x} - \frac{y(x_0)}{x_0} = \int_{x_0}^{x} e^s ds$$

$$\Rightarrow y(x) = \frac{x\, y(x_0)}{x_0} + x\int_{x_0}^{x} e^s ds$$

$$e^{\int p(x)dx} y'(x) + y(x) e^{\int p(x)dx} p(x) = e^{\int p(x)dx} q(x)$$

$$\left(y(x) e^{\int p(x)dx}\right)' = e^{\int p(x)dx} q(x)$$

$$y(x) e^{\int p(x)dx} = c + \int e^{\int p(x)dx} q(x)dx$$

$$\boxed{y(x) = e^{-\int p(x)dx}\left[c + \int e^{\int p(x)dx} q(x)dx\right]}$$

με ορισμένη ολοκλήρωση:

$$e^{\int_{x_0}^{x} p(s)ds} y'(x) + y(x) e^{\int_{x_0}^{x} p(s)ds} p(x) = e^{\int_{x_0}^{x} p(s)ds} q(x), \quad x \in I$$

$$\left(y(x) e^{\int_{x_0}^{x} p(s)ds}\right)' = e^{\int_{x_0}^{x} p(s)ds} q(x), \quad x \in I$$

$$e^{\int_{x_0}^{x} p(s)ds} y(x) - e^{\int_{x_0}^{x_0} p(s)ds} y(x_0) = \int_{x_0}^{x} q(u) e^{\int_{x_0}^{u} p(s)ds} du$$

$$e^{\int_{x_0}^{x} p(s)ds} y(x) = y(x_0) + \int_{x_0}^{x} q(u) e^{\int_{x_0}^{u} p(s)ds} du$$

$$\boxed{y(x) = e^{-\int_{x_0}^{x} p(s)ds}\left[y(x_0) + \int_{x_0}^{x} q(u) e^{\int_{x_0}^{u} p(s)ds} du\right]}$$

<u>Παράδειγμα 2 σελ 31</u>

$$xy' - 2y = x^2, \quad y(1) = 0, \quad I = (0, +\infty)$$

~~$y(x) = e$~~ $$y' - \frac{2}{x} y = x$$

$$\Rightarrow y(x) = e^{-\int_1^x (-\frac{2}{s})ds}\left[y(1) + \int_1^x s e^{\int_1^s (-\frac{2}{u})du} ds\right]$$

$$y(x) = e^{2\ln s|_1^x}\left[0 + \int_1^x s e^{-2\ln u|_1^s} ds\right]$$

<u>A14(ii) σελ 56</u>

$$y' + y = \frac{1}{1+x^2}, \quad y(1) = 2$$

$$y(x) = e^{-\int_1^x 1 ds}\left[y(1) + \int_1^x \frac{1}{1+s^2} e^{\int_1^s 1 du} ds\right], \quad x \in R$$

$$y(x) = e^{-x+1}\left[y(1) + \int_1^x \frac{1}{1+s^2} e^{s-1} ds\right]$$

$$y(x) = e^{-x+x_0}\left[y(x_0) + \int_{x_0}^{x} \frac{1}{1+s^2} e^{s-x_0} ds\right] = e^{x_0} y(x_0) \cdot e^{-x} + \frac{e^{x_0} e^{-x}}{e^{x_0}} \int_{x_0}^{x} \frac{e^s}{1+s^2} ds$$

$$\lim_{x \to \infty} y(x) = ?$$

Άσκηση

$ay' + by = ke^{-\lambda x}$, $a,b,k>0$, $\lambda \ge 0$

i) $\lambda = 0$ νδο $\lim_{+\infty} y(x) = k/b$

ii) $\lambda>0$ νδο $\lim_{+\infty} y(x) = 0$

$y' + \frac{b}{a} y = \frac{k}{a} e^{-\lambda x}$, $x\in\mathbb{R}$

$y(x) = e^{-\int_0^x \frac{b}{a} ds}\left[y(0) + \int_0^x \frac{k}{a} e^{-\lambda s} e^{\int_0^s \frac{b}{a} du} ds\right]$, $x\in\mathbb{R}$

$$y(x) = e^{-\frac{b}{a}x}\left[y(0) + \frac{k}{a}\int_0^x e^{-\lambda s} e^{(b/a)s} ds\right]$$

$$y(x) = \underset{\downarrow\, 0}{y(0)} e^{-\frac{b}{a}x} + e^{-\frac{b}{a}x}\cdot\frac{k}{a}\begin{cases} x, & b/a=\lambda \\ \frac{1}{\frac{b}{a}-\lambda}\left(e^{(\frac{b}{a}-\lambda)x} - 1\right), & b/a\ne\lambda\end{cases}$$

$= y(0)e^{-\frac{b}{a}x} + \frac{k}{a}e^{-\frac{b}{a}x}\cdot x$ ή

ή $y(0)e^{-\frac{b}{a}x} + \frac{1}{\frac{b}{a}-\lambda}\left(e^{-\lambda x} - e^{-\frac{b}{a}x}\right]$

Άσκηση 1.43

$y'+y = g(x)$, $y(0)=0$ | $g(x) = \begin{cases}2, & x\in[0,1]\\ 0, & x>1\end{cases}$

$y(x) = e^{-\int_0^x 1 ds}\left[\overset{0}{y(0)} + \int_0^x g(s) e^{\int_1^s 1 du} ds\right]$, $x\ge 0$

i) $0\le x\le 1$: $y(x) = e^{-x}\int_0^x 2e^s ds = e^{-x}\, 2(e^x-1)$, $x\in[0,1]$

$y(x) = 2-2e^{-x}$, $x\in[0,1]$

ii) $x>1$: $y(x) = e^{-x}\left[0 + \int_0^x g(s) e^{\int_1^s 1 du} ds\right]$

$= e^{-x}\left(\int_0^1 g(s)e^s ds + \int_1^x \cancel{0e^s} ds\right)$

$= e^{-x}\int_0^1 2e^s ds = \ldots$

Άσκηση

$y' + by = \sin(ax)$, $a, b \neq 0$, $b > 0$

$\lim_{+\infty} y(x)$ αν υπάρχει

$$y(x) = e^{-\int_0^x b\,ds}\left[y(0) + \int_0^x \sin(as)\, e^{\int_0^s b\,du}\, ds\right]$$

$$= y(0)e^{-bx} + e^{-bx}\int_0^x \sin(as)\, e^{bs}\, ds \quad , x \in \mathbb{R}$$

$$= y(0)\underset{\substack{\downarrow \\ 0}}{e^{-bx}} + e^{-bx}\frac{1}{a^2+b^2}(b\sin ax - a\cos ax) = M\sin(ax+\theta)$$

$$k\cos x + \lambda \sin x = \sqrt{k^2+\lambda^2}\left(\frac{k}{\sqrt{k^2+\lambda^2}}\cos x + \frac{\lambda}{\sqrt{k^2+\lambda^2}}\sin x\right) = \sqrt{k^2+\lambda^2}\,(\sin\theta\cos x + \cos\theta\sin x)$$

$$= \sqrt{k^2+\lambda^2}\,\sin(\theta + x)$$

$$\sin\theta = \frac{k}{\sqrt{k^2+\lambda^2}} \quad , \cos\theta = \frac{\lambda}{\sqrt{k^2+\lambda^2}}$$

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$y' + py = q \qquad p, q \in C[0,\infty)$

$\exists x_0 \geq 0 , \exists \mu > 0 : p(x) \geq \mu \quad \forall x \geq x_0$

$\lim_{+\infty} q(x) = 0$

$\leadsto \lim_{+\infty} y(x) = 0$

$$y(x) = e^{-\int_{x_0}^x p(s)ds}\left[y(x_0) + \int_{x_0}^x q(s)\, e^{\int_{x_0}^s p(u)du}\, ds\right]$$

$$y(x) = \underbrace{e^{-\int_{x_0}^x p(s)ds}\, y(x_0)}_{(A)} + \underbrace{e^{-\int_{x_0}^x p(s)ds}\int_{x_0}^x q(s)\, e^{\int_{x_0}^s p(u)du}\, ds}_{(B)}$$

(A): $\left.\begin{array}{l} x_0 \leq s \leq x \\ \mu \leq p(x) \end{array}\right| \Rightarrow e^{-\int_{x_0}^x p(s)ds} \leq e^{-\mu(x-x_0)}$

$\mu(x-x_0) = \int_{x_0}^x \mu \leq \int_{x_0}^x p(s)ds$

$$0 \leq e^{-\int_{x_0}^x p(s)ds} \leq e^{\mu x_0}\underset{\substack{\downarrow \\ 0}}{e^{-\mu x}}$$

(B): $$\frac{\int_{x_0}^x q(s)\, e^{\int_{x_0}^s p(u)du}\, ds}{e^{\int_{x_0}^x p(s)ds}}$$